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# STOCHASTIC FUNCTIONAL EXPANSION IN ELASTICITY OF HETEROGENEOUS SOLIDS

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**Abstract**—The Volterra–Wiener functional expansion is employed to the analysis of statistic properties for random heterogeneous solids. For simplicity, the technique is displayed on an elastic suspension of spheres. The basis function in the expansion is chosen as that corresponding to the so-called "perfect disorder" of spheres (PDS), recently introduced by the authors. An infinite hierarchy of equations for the kernels in the expansion is derived whose truncating after the $n$th equation is shown to yield results for the averaged statistical characteristics which are valid to order $c_f^n$, where $c_f$ is the volume fraction of the spheres. The kernels for the first and the second approximations, $n = 1, 2$, are found and related to the displacement fields in an infinite elastic body containing, respectively, one and two spherical inhomogeneities. Within the frame of the so-called singular approximation the overall tensor of elastic moduli for a suspension of perfectly disordered spheres is shown to coincide to the order $c_f^2$ with a formula, earlier obtained by means of the method of the effective field.

## 1. INTRODUCTION

In this paper we consider the statistical problem of specifying the random fields—displacement, strain, etc.—in a heterogeneous linear elastic material whose properties vary randomly with position. Although the technique employed has much wider range of applicability for such materials, we shall confine the study, for simplicity, to a random elastic suspension of spheres. In particular we shall pay a special attention to the overall (effective) elastic properties of the material.

As known, the problem of predicting the properties of heterogeneous solids on the base of a certain information concerning their microstructure has enjoyed a considerable interest due, e.g. to its importance in mechanics of composite materials, cf. [1–3] for detailed and comprehensive surveys of the existing studies from various viewpoints. These surveys clearly indicate that a great amount of the work done suffices with models that lead to certain relations for the overall properties. These models are not extracted, however, from a rigorous statistical analysis, so that it is never clear, as a rule, what kind of random constitution, if any, lies behind the proposed relations for the overall properties. On the other hand, as seen from[1–3], the only rigorous and fruitful procedures in mechanics of heterogeneous materials are still those described in detail by M. Beran([4], Chap. 5), i.e. the perturbation and variational ones.

In perturbation method, it is assumed that the constituents have slightly different material properties. In this case the hierarchy of statistical moment equations can be truncated in an obvious manner([4], p. 222). In the real heterogeneous materials, the differences in the constituent properties are, however, considerable so that the perturbation method fails, as a rule, to bring reasonable results.

Let us note now that for a two-phase particulate material there is always a parameter which does not exceed 1—this is the volume fraction $c_f$ of the discrete (filler) constituent; in fact $c_f$ rarely exceeds 0.4. That is why a counterpart of the perturbation method for which the solution is to be obtained in a virial form, i.e. as a power series with respect to $c_f$, should have much greater range of application for heterogeneous materials than the perturbation technique. (Prior now, only the first-order term in the virial expansion for the overall moduli has been obtained in a number of cases; this corresponds to a dilute suspension of particles in a matrix. Certain work has been also devoted to estimating the second-order terms for a suspension of sheres; see, e.g. [5–7].) As far as the aim of the full statistical solution to the problem, the basic difficulty to be surmounted when devising such a virial method is the derivation of an infinite hierarchy of equations—a counterpart of the said hierarchy of moment equations—whose truncation assures results for the averaged statistical characteristics which hold to order $c_f^n$, for any $n$ prescribed.

In this paper it is shown, making use of the previous works[8–11] of the authors, that a hierarchy of this kind can be derived by means of the Volterra–Wiener functional expansion. This is illustrated in the case of a special distribution of equisized spherical inclusions in an infinite elastic medium, which seems to be of particular interest, namely the so-called "perfect disorder" of spheres[10]. The equations which describe the full statistical solution up to the order $c_f^2$ are derived and related to those for a body containing one or two spherical inhomogeneities. Within the frame of the so-called singular approximation[12], the overall tensor of elastic moduli is calculated explicitly.

## 2. VOLTERRA–WIENER EXPANSION

Consider an infinite medium which consists of an array of spheres, each of radius $a$ and with tensor of elastic moduli $\mathbf{L}_f$, randomly distributed in a matrix with tensor of elastic moduli $\mathbf{L}_m$. The volume fractions of the matrix and of the spheres are, respectively, $c_m$ and $c_f$, so that $c_m + c_f = 1$. Hereafter we supply all the quantities connected to the matrix with the subscript "$m$" and those connected to the inclusions (the filler)—with "$f$".

Let $\mathbf{x}_j$ be the random system of points which are centers of the spheres. As their radius is constant, in order to obtain an exhaustive description of the random suspension under consideration, it is necessary to prescribe only the statistical properties of the random system $\mathbf{x}_j$. Such a description is provided by the multipoint distribution functions $f_n(\mathbf{y}_1, \ldots, \mathbf{y}_n)$, cf. e.g. [13], which give the probability $\mathrm{d}P$ to find simultaneously a point from $\mathbf{x}_j$ per each of the infinitesimal volumes $\mathbf{y}_i < \mathbf{y} < \mathbf{y}_i + \mathrm{d}^3\mathbf{y}_i$ to be

$$\mathrm{d}P = f_n(\mathbf{y}_1, \ldots, \mathbf{y}_n)\, \mathrm{d}^3\mathbf{y}_1 \ldots \mathrm{d}^3\mathbf{y}_n. \tag{2.1}$$

The functions $f_n$ are symmetric functions of their arguments; they are to satisfy, in particular, the conditions

$$\frac{1}{|D|^n} \int_D \cdots \int_D f_n(\mathbf{y}_1, \ldots, \mathbf{y}_n)\, \mathrm{d}^3\mathbf{y}_1 \ldots \mathrm{d}^3\mathbf{y}_n \xrightarrow[D \to \mathbb{R}^3]{} \gamma^n,$$

where $\gamma$ is the mean number of points per unit volume, so that $c_f = \frac{4}{3}\pi a^3\gamma$; $|D| =$ volume $(D)$.

We assume the system $\mathbf{x}_j$ to be statistically homogeneous, then $f_n(\mathbf{y}_1, \ldots, \mathbf{y}_n) = f'_n(\mathbf{y}_2 - \mathbf{y}_1, \ldots, \mathbf{y}_n - \mathbf{y}_1)$; the prime will be omitted in what follows. In particular, $f_1 \equiv \gamma$.

It is important to mention that for the Poisson system of points (or the Poisson pattern, cf. [4], p. 207), the multipoint distribution functions, according to [13], p. 143, are

$$f_n = \gamma^n, \qquad n = 1, 2, \ldots. \tag{2.2}$$

With the random system $\mathbf{x}_j$ given, we introduce after[13], p. 140, the random field

$$\psi(\mathbf{x}) = \sum_j \delta(\mathbf{x} - \mathbf{x}_j), \tag{2.3}$$

where $\delta$ is the Dirac delta function. The random field of tensor of elastic moduli, $\mathbf{L}(\mathbf{x})$, for the medium under study has a simple representation by means of $\psi(\mathbf{x})$, namely

$$\mathbf{L}(\mathbf{x}) = \langle \mathbf{L} \rangle + [\mathbf{L}] \int h(\mathbf{x} - \mathbf{y})\, C_{\psi}^{(1)}(\mathbf{y})\, d^3\mathbf{y}, \tag{2.4}$$

where

$$C_{\psi}^{(1)}(\mathbf{y}) = \psi(\mathbf{y}) - \gamma \tag{2.5}$$

is the centered random field eqn (2.3); $\langle \mathbf{L} \rangle = c_m \mathbf{L}_m + c_f \mathbf{L}_f$, $[\mathbf{L}] = \mathbf{L}_f - \mathbf{L}_m$ and $h(\mathbf{x}) = 1$ for $|\mathbf{x}| \leq 1$, and vanishes otherwise.

The displacement field, $\mathbf{u}(\mathbf{x})$, in the random medium is governed by the equations (in the case of statics and at vanishing body forces)

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{x}) = 0, \qquad \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{L}(\mathbf{x}) : \boldsymbol{\epsilon}(\mathbf{x}), \qquad \boldsymbol{\epsilon}(\mathbf{x}) = \tfrac{1}{2}(\nabla \mathbf{u} + \mathbf{u} \nabla), \tag{2.6}$$

so that $\boldsymbol{\epsilon}(\mathbf{x})$ and $\boldsymbol{\sigma}(\mathbf{x})$ are, respectively, the strain and stress tensors at the point $\mathbf{x}$; the colon denotes contraction of the tensors with respect to two pairs of indexes. We prescribe the mean value of the strain tensor, $\mathbf{E}$, to be constant

$$\langle \boldsymbol{\epsilon}(\mathbf{x}) \rangle = \mathbf{E} \tag{2.7}$$

which will play the role of a boundary condition for the random eqns (2.6). The brackets $\langle \cdot \rangle$ hereafter denote ensemble averaging, cf. [4].

The eqns (2.6) together with (2.7) define a nonlinear operator which transforms the random field of coefficients, $\mathbf{L}(\mathbf{x})$—the "input," into the random displacement field, $\mathbf{u}(\mathbf{x})$—the "output." As first noted in [10–11], a general and powerful method for a successful attack of stochastic problems for random heterogeneous materials consists in expanding this operator into a Volterra–Wiener series, generated by the input $\mathbf{L}(\mathbf{x})$, and truncating this series afterward in order to get approximate models or solutions. However, the usefulness of the Volterra–Wiener series for a given problem depends to a great extent, as it was acknowledged by Wiener himself, on the possibility of rendering the series orthogonal in stochastic sense. For an arbitrary random field $\mathbf{L}(\mathbf{x})$ there is no algorithm how to construct such orthogonal series; moreover, it is not clear whether they exist at all.

If the input is the white Gaussian noise, the orthogonal Volterra–Wiener series were constructed by Wiener himself[14]; they appeared to be generated by the multivariate Hermite polynomials so that the corresponding series was called Wiener–Hermite expansion. This expansion was employed to various physical problems, such as system identification, turbulence, etc., cf. [15] for a detailed survey. As argued in [9–10], in problems concerned with heterogeneous materials, especially with those of a particulate type, the input is a point-random function, and thus it differs essentially from the continuous in stochastic sense Gaussian white noise. Moreover, if no detailed information about the distribution of the particles is available (a typical case we are faced with in reality), we can think their locations statistically independent which immediately brings into view the Poisson random system as the most suitable basis function.

First to replace the Gaussian white noise by the Poisson random system in Volterra–Wiener expansion was Ogura[16], who constructed the orthogonal Wiener functionals (by means of the Charlier polynominals, as it appeared), and named the respective series Poisson–Wiener expansion. He did so, however, only on a formal basis and without any applications. Only recently Christov[9, 17] revealed the spectacular performance of the Poisson–Wiener expansion for nonlinear stochastic systems and developed the necessary technique with application to the model case of Burgers turbulence and to a stochastic problem for a Poiseuille flow.

A straightforward application of the Poisson–Wiener expansion to heterogeneous materials is, however, impossible due to the fact that the inhomogeneities possess finite size. That is why we introduce, after [10], a Poissonian-like system of random points for which the points are only correlated in such a manner that their appearance arbitrary close one to another is forbidden in order to prevent overlapping of the inhomogeneities. For the suspension of spheres under consideration the latter means, in the language of the multipoint distribution functions (2.1), that

$$f_n(\mathbf{y}_1, \ldots, \mathbf{y}_n) = \gamma^n \prod_{\substack{i,j=1 \\ i\neq j}}^{n} Q(\mathbf{y}_i - \mathbf{y}_j), \tag{2.8}$$

$$Q(\mathbf{y}) = 1 - R(\mathbf{y}), \qquad R(\mathbf{y}) = \begin{cases} 1, & |\mathbf{y}| < 2a, \\ 0, & |\mathbf{y}| \geq 2a, \end{cases} \tag{2.9}$$

$n = 2, 3, \ldots$. As argued in [10], the functions (2.8) describe a random system of points which serve as centers of a perfectly disordered system of equisized spheres of radius $a$. That is why a random system of points $\mathbf{x}_j$, whose multipoint distribution functions are given by eqn (2.8), was called by the authors[10] a perfect disorder of spheres or, for brevity, a PDS-field.

To clarify a bit more such a terminology, let us note, first of all, that the cumulants (or semi-invariants) of the random field (2.3) generated by a Poisson system of points are Dirac's delta functions ([13], p. 141). As the cumulants are measures of statistical dependence between random variables ([13], p. 18), see also [18], the delta-shape of the cumulants is a formal expression of the fact that there are no statistical connections between the location of the points in a Poisson system. In this sense, the Poisson random system can be also called perfect disorder of points. It should be pointed out here that the term, "perfect disorder," was first coined by Kröner[19] whose aim was to introduce a certain idealized random medium for which there is no statistical connection between the material properties at its different points. However, Kröner took the respective moments, instead of cumulants, to be $\delta$-functions, and also he did not account for the finite size of the inhomogeneities. As a result the original Kröner's notion of perfect disorder was later critisized, cf. [2], p. 14. Acknowledging this critisism, Kröner modified his definition, see, e.g. [20]. (Let us note in passing, in authors' view, this modification lacks the physical clarity of the original Kröner's idea; moreover, it seems well adapted to polycrystals only, and not to two-phase materials whose particulate phase possesses distinguishable shape, cf. [21], p. 219.) The above introduced PDS-field arises, in a sense, from a line of thinking close to that of Kröner of 1967, [19], when adding two crucial improvements to his definition of perfect disorder: (1) replacing moments by cumulants to be $\delta$-functions, in order to make the definition statistically correct, and (2) taking account of the finite size of the inhomogeneities.

The formal theory of the Volterra–Wiener expansion with the PDS-field as a basis function is built up in detail in [10], and the basic results obtained are the following: The orthogonal Wiener functionals for this case appear to be generated, similarly to the Poisson–Wiener expansion, by the multivariate Charlier polynomials $C_\psi^{(n)}$ of the random variable $\psi$, cf. (2.3), generated by the PDS-field $\mathbf{x}_j$. Thus, the random displacement field, $\mathbf{u}(\mathbf{x})$, which solves the eqns (2.6) and (2.7), can be expanded into the orthogonal functional series

$$\mathbf{u}(\mathbf{x}) = \mathbf{E}\cdot\mathbf{x} + \int \mathbf{T}_1(\mathbf{x} - \mathbf{y})\, C_\psi^{(1)}(\mathbf{y})\, d^3\mathbf{y} + \iint \mathbf{T}_2(\mathbf{x} - \mathbf{y}_1, \mathbf{x} - \mathbf{y}_2)\, C_\psi^{(2)}(\mathbf{y}_1, \mathbf{y}_2)\, d^3\mathbf{y}_1\, d^3\mathbf{y}_2 + \cdots \tag{2.10}$$

with nonrandom kernels $\mathbf{T}_1$, $\mathbf{T}_2$, etc. (Hereafter, if the integration domain is not explicitly indicated, the integrals are taken over the whole $\mathbb{R}^3$.) The kernels $\mathbf{T}_n$, $n \geq 2$, however, should comply with the condition

$$\mathbf{T}_n(\mathbf{z}_1, \ldots, \mathbf{z}_n) = 0, \tag{2.11}$$

if there exists at least one pair of indexes $i, j$, for which $|\mathbf{z}_i - \mathbf{z}_j| < 2a$. (This condition is fully natural and always can be adopted for the suspension of spheres under consideration, since we never have two points of the system $\mathbf{x}_j$ situated closer than the spheres' diameter $2a$.) The expansion (2.10), under the condition (2.11) is a virial one, in the sense that the $n$th-order term contributes to the average characteristics of the field $\mathbf{u}(\mathbf{x})$ quantities of order $c_f^n$.

The first two Charlier polynomials are

$$C_\psi^{(1)}(\mathbf{y}) = \psi(\mathbf{y}) - \gamma,$$
$$C_\psi^{(2)}(\mathbf{y}_1, \mathbf{y}_2) = \psi(\mathbf{y}_1)\psi(\mathbf{y}_2) - \delta(\mathbf{y}_1 - \mathbf{y}_2)\psi(\mathbf{y}_2) - \gamma(\psi(\mathbf{y}_1) + \psi(\mathbf{y}_2)) + \gamma^2, \tag{2.12}$$

cf. [16, 9]. [Note that we already came across the first Charlier polynomial, cf. eqn (2.5).] The first few moments of these polynomials, which will be needed below, are

$$\begin{aligned}
\langle C_\psi^{(1)}(\mathbf{y})\rangle = 0, \qquad \langle C_\psi^{(1)}(\mathbf{y}_1)C_\psi^{(1)}(\mathbf{y}_2)\rangle &= \gamma\,\delta(\mathbf{y}_{12}) - \gamma^2 R(\mathbf{y}_{12}),\\
\langle C_\psi^{(1)}(\mathbf{y}_1)C_\psi^{(1)}(\mathbf{y}_2)C_\psi^{(1)}(\mathbf{y}_3)\rangle &= \gamma\,\delta(\mathbf{y}_{12})\,\delta(\mathbf{y}_{13}) - \gamma^2\,3\{\delta(\mathbf{y}_{12})\,R(\mathbf{y}_{23})\}_s\\
&\quad + \gamma^3(3\{R(\mathbf{y}_{12})\,R(\mathbf{y}_{23})\}_s - R(\mathbf{y}_{12})\,R(\mathbf{y}_{23})\,R(\mathbf{y}_{31})),\\
\langle C_\psi^{(1)}(\mathbf{y}_1)C_\psi^{(1)}(\mathbf{y}_2)C_\psi^{(2)}(\mathbf{y}_3, \mathbf{y}_4)\rangle &= \gamma^2(1 - R(\mathbf{y}_{12}))[\delta(\mathbf{y}_{13})\,\delta(\mathbf{y}_{24})\\
&\quad + \delta(\mathbf{y}_{14})\delta(\mathbf{y}_{23})] + o(\gamma^2),\\
\langle C_\psi^{(2)}(\mathbf{y}_1, \mathbf{y}_2)C_\psi^{(2)}(\mathbf{y}_3, \mathbf{y}_4)\rangle &= \gamma^2(1 - R(\mathbf{y}_{12})[\delta(\mathbf{y}_{13})\delta(\mathbf{y}_{24}) + \delta(\mathbf{y}_{14})\delta(\mathbf{y}_{23})] + o(\gamma^2),\\
\langle C_\psi^{(2)}(\mathbf{y}_1, \mathbf{y}_2)C_\psi^{(2)}(\mathbf{y}_3, \mathbf{y}_4)C_\psi^{(1)}(\mathbf{y}_5)\rangle &= \gamma^2(1 - R(\mathbf{y}_{12})[\delta(\mathbf{y}_{15}) + \delta(\mathbf{y}_{25})][\delta(\mathbf{y}_{13})\delta(\mathbf{y}_{24})\\
&\quad + \delta(\mathbf{y}_{14})\delta(\mathbf{y}_{23})] + o(\gamma^2).
\end{aligned} \tag{2.13}$$

Here $\mathbf{y}_{ij} = \mathbf{y}_i - \mathbf{y}_j$ and $\{\cdot\}_s$ denotes symmetrization with respect to the indexes listed in the brackets, e.g. $3\{R(\mathbf{y}_{12})R(\mathbf{y}_{23})\}_s = R(\mathbf{y}_{12})R(\mathbf{y}_{23}) + R(\mathbf{y}_{23})R(\mathbf{y}_{31}) + R(\mathbf{y}_{31})R(\mathbf{y}_{12})$, the factor in front of the brackets indicates how many terms enter the full expression.

It should be pointed out that the above formulas (2.13) are particular cases of more general relations for the third moments of the Charlier polynomials, which were first derived in [9] and which appeared to be of crucial importance when employing the Poisson–Wiener expansion to nonlinear stochastic problems, cf. [9, 10] for more details.

Upon substituting eqns (2.4) and (2.10) into the Hooke law, we find now the stress field in the material to be

$$\begin{aligned}
\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{L}(\mathbf{x}):\boldsymbol{\epsilon}(\mathbf{x}) &= \langle\mathbf{L}\rangle:\mathbf{E} + \langle\mathbf{L}\rangle: \int \operatorname{def}\mathbf{T}_1(\mathbf{x} - \mathbf{y})C_\psi^{(1)}(\mathbf{y})\,d^3\mathbf{y}\\
&\quad + \cdots + [\mathbf{L}]:\mathbf{E}\int h(\mathbf{x} - \mathbf{y})C_\psi^{(1)}(\mathbf{y})\,d^3\mathbf{y}\\
&\quad + [\mathbf{L}]: \iint \operatorname{def}\mathbf{T}_1(\mathbf{x} - \mathbf{y}_1)h(\mathbf{x} - \mathbf{y}_2)C_\psi^{(1)}(\mathbf{y}_1)C_\psi^{(1)}(\mathbf{y}_2)\,d^3\mathbf{y}_1\,d^3\mathbf{y}_2\\
&\quad + \cdots,
\end{aligned} \tag{2.14}$$

where $\operatorname{def}\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + \mathbf{v}\nabla)$. Upon averaging eqn (2.14) and using the mentioned orthogonality of the terms of different order in the expansion (2.10), we obtain for the overall tensor of the elastic moduli, $\mathbf{L}^*$, of the suspension under study

$$\langle \boldsymbol{\sigma}(\mathbf{x}) \rangle = \mathbf{L}^* : \langle \boldsymbol{\epsilon}(\mathbf{x}) \rangle = \mathbf{L}^* : \mathbf{E}$$

$$= \langle \mathbf{L} \rangle : \mathbf{E} + [\mathbf{L}] : \iint \operatorname{def} \mathbf{T}_1(\mathbf{x} - \mathbf{y}_1) h(\mathbf{x} - \mathbf{y}_2) \langle C_\psi^{(1)}(\mathbf{y}_1) C_\psi^{(1)}(\mathbf{y}_2) \rangle \, \mathrm{d}^3\mathbf{y}_1 \, \mathrm{d}^3\mathbf{y}_2. \quad (2.15)$$

Thus the tensor $\mathbf{L}^*$ depends on the first kernel $\mathbf{T}_1$ only. Moreover, making use of eqns (2.13), we can rewrite (2.15) as follows

$$\mathbf{L}^* : \mathbf{E} = \langle \mathbf{L} \rangle : \mathbf{E} + \gamma [\mathbf{L}] : \int_{V_a} \operatorname{def} \mathbf{S}(\mathbf{x}) \, \mathrm{d}^3\mathbf{x}, \quad (2.16)$$

$$\mathbf{S}(\mathbf{x}) = \mathbf{T}_1(\mathbf{x}) - \gamma \int R(\mathbf{x} - \mathbf{y}) \, \mathbf{T}_1(\mathbf{y}) \, \mathrm{d}^3\mathbf{y}. \quad (2.17)$$

Therefore, one needs to know only the values of $\mathbf{S}(\mathbf{x})$ within the sphere $\mathbf{V}_a = \{\mathbf{x};\, |\,\mathbf{x}\,| \le a\}$ in order to calculate the overall elastic properties of the suspension of perfectly disordered spheres. [These values, however, depend in general on the whole solution to the stochastic problem (2.6)–(2.7).]

To get equations for specifying the kernels $\mathbf{T}_1, \mathbf{T}_2$, etc., we insert the representation (2.14) of the stress field into the equilibrium equation (2.6), multiply by the functions

$$\tilde{g}_1 = C_\psi^{(1)}(\mathbf{0}), \qquad \tilde{g}_2 = (1 - R(\mathbf{z})) C_\psi^{(2)}(\mathbf{0}, \mathbf{z}), \; \ldots \, ,$$

as proposed in [9–10], and average the results. This procedure brings forth an infinite hierarchy of conjugated equations for the kernels in the expansion (2.10); the $n$th equation contains three consecutive kernels $\mathbf{T}_{n-1}$, $\mathbf{T}_n$, $\mathbf{T}_{n+1}$. This hierarchy is similar to that derived in [10] for the problem of heat conduction through the suspension under consideration. The basic result here is that, due to the virial character of the expansion (2.10), the hierarchy can be splitted in an obvious manner if the kernels are looked for as power series

$$\mathbf{T}_n(\mathbf{x}) = \sum_{k=0}^{\infty} \gamma^k \, \mathbf{T}_{nk}(\mathbf{x}), \qquad n = 1, 2, \ldots . \quad (2.18)$$

Thus, for an elastic suspension of perfectly disordered spheres, we can solve, at least in principle, the stochastic problem (2.6)–(2.7) with an order of accuracy $\gamma^n$, i.e. $c_f^n$, for each $n$ prescribed, and the obtained solution will be a full-scale statistical one, in the sense that we shall know not only the overall tensor $\mathbf{L}^*$, but also all the moments like $\langle \boldsymbol{\epsilon}(\mathbf{y}_1) \otimes \boldsymbol{\epsilon}(\mathbf{y}_2) \rangle$, $\langle \boldsymbol{\epsilon}(\mathbf{y}_1) \otimes \boldsymbol{\sigma}(\mathbf{y}_2) \rangle$, etc., to the same order $c_f^n$.

To illustrate the above said we shall consider in detail the first- and the second-order approximations. They will correspond to retaining only the first or the first two terms in the expansion (2.10), and will bring along solutions to the considered stochastic problem which will hold to order $c_f$ and $c_f^2$, respectively. For that purpose we shall need only the first two equations of the said hierarchy truncated to order $\gamma^2$; they are easily derived by means of eqns (2.13) in the above indicated manner:

$$\nabla \cdot \{ \langle \mathbf{L} \rangle : (\gamma \mathbf{Q}_1(\mathbf{x}) - \gamma^2 \int \mathbf{Q}_1(\mathbf{x} - \mathbf{y}) R(\mathbf{y}) \, \mathrm{d}^3\mathbf{y})$$

$$+ [\mathbf{L}] : \mathbf{E} (\gamma h(\mathbf{x}) - \gamma^2 \int h(\mathbf{x} - \mathbf{y}) R(\mathbf{y}) \, \mathrm{d}^3\mathbf{y})$$

$$+ [\mathbf{L}] : (\mathbf{Q}_1(\mathbf{x}) \, \gamma h(\mathbf{x}) - \gamma^2 \int \mathbf{Q}_1(\mathbf{x} - \mathbf{y}) h(\mathbf{x} - \mathbf{y}) R(\mathbf{y}) \, \mathrm{d}^3\mathbf{y}$$

$$- \gamma^2 \mathbf{Q}_1(\mathbf{x}) \int h(\mathbf{x} - \mathbf{y}) R(\mathbf{y}) \, \mathrm{d}^3\mathbf{y} - \gamma^2 h(\mathbf{x}) \int \mathbf{Q}_1(\mathbf{x} - \mathbf{y}) R(\mathbf{y}) \, \mathrm{d}^3\mathbf{y}$$

$$+ \gamma^3 \iint \mathbf{Q}_1(\mathbf{x} - \mathbf{y}_1)h(\mathbf{x} - \mathbf{y}_2)[R(\mathbf{y}_1 - \mathbf{y}_2)R(\mathbf{y}_2) + R(\mathbf{y}_1)R(\mathbf{y}_2)$$
$$+ R(\mathbf{y}_1)R(\mathbf{y}_1 - \mathbf{y}_2) - R(\mathbf{y}_1)R(\mathbf{y}_2)R(\mathbf{y}_1 - \mathbf{y}_2)]\, d^3\mathbf{y}_1\, d^3\mathbf{y}_2)$$
$$+ 2\gamma^2[\mathbf{L}]: \int \mathbf{Q}_2(\mathbf{x} - \mathbf{y}, \mathbf{x})h(\mathbf{x} - \mathbf{y})\, d^3\mathbf{y}\} + o(\gamma^2; \mathbf{T}_2) = 0; \quad (2.19)$$

$$\nabla\cdot\{2\gamma^2(\langle\mathbf{L}\rangle + (h(\mathbf{x}) + h(\mathbf{x} - \mathbf{z}))[\mathbf{L}]):\mathbf{Q}_2(\mathbf{x} - \mathbf{z}, \mathbf{x})$$
$$+ \gamma^2[\mathbf{L}]:(\mathbf{Q}_1(\mathbf{x} - \mathbf{z})h(\mathbf{x}) + \mathbf{Q}_1(\mathbf{x})h(\mathbf{x} - \mathbf{z}))\} + o(\gamma^2) = 0, \quad (2.20)$$

where $\mathbf{Q}_i(\mathbf{x}) = \text{def}\ \mathbf{T}_i(\mathbf{x})$, $i = 1, 2$. Due to reasons which will become clear in Section 5, we retained in eqn (2.19) all terms generated by the first kernel $\mathbf{T}_1$, while rejecting the terms of order higher than $\gamma^2$ introduced by $\mathbf{T}_2$.

## 3. FIRST-ORDER APPROXIMATION

As follows from eqn (2.19), to get a solution to the problem (2.6)–(2.7) which holds to order $\gamma$, i.e. $c_f$, we have to retain in the virial expansion (2.18) for $\mathbf{T}_1(\mathbf{x})$ only the zeroth-order term

$$\mathbf{T}_1(\mathbf{x}) = \mathbf{T}_{10}(\mathbf{x}). \quad (3.1)$$

The latter is governed by the linear in $\gamma$ version of eqn (2.19):

$$\nabla\cdot\{\mathbf{L}_m:\mathbf{Q}_{10}(\mathbf{x}) + h(\mathbf{x})[\mathbf{L}]:(\mathbf{E} + \mathbf{Q}_{10}(\mathbf{x}))\} = 0, \quad (3.2)$$

$\mathbf{Q}_{10}(\mathbf{x}) = \text{def}\ \mathbf{T}_{10}(\mathbf{x})$. This is nothing but the equation for the disturbance to the strain field in the unbounded matrix, introduced by a single spherical inclusion, provided that the strain tensor at infinity has a constant prescribed value $\mathbf{E}$. As known, the solution to this problem, $\mathbf{Q}_{10}(\mathbf{x})$, is constant within the inclusion[22], so that

$$\mathbf{E}_f = \mathbf{E} + \mathbf{Q}_{10}(\mathbf{x}) = \mathbf{A}(\mathbf{L}_m, \mathbf{L}_f):\mathbf{E}, \qquad |\mathbf{x}| < a. \quad (3.3)$$

Thus $\mathbf{A}(\mathbf{L}_m, \mathbf{L}_f)$ is the forth-rank tensor which transforms the homogeneous strain, $\mathbf{E}$, applied to the matrix at infinity, into the homogeneous strain $\mathbf{E}_f$ which makes appearance into a single spherical inclusion embedded into the matrix.

For the effective tensor of elastic moduli we have now, due to eqns (2.16), (2.17), and (3.3), the desired first-order relation

$$\mathbf{L}^* = \mathbf{L}_m + c_f[\mathbf{L}]:\mathbf{A}(\mathbf{L}_m, \mathbf{L}_f) + o(c_f). \quad (3.4)$$

If both matrix and inclusions are isotropic, the tensor $\mathbf{A}$ is well known[23]; the overall bulk and shear moduli of the suspension then become, respectively,

$$\frac{k^*}{k_m} = 1 + \frac{[k]}{k_m + \alpha_m[k]} c_f + o(c_f),$$
$$\frac{\mu^*}{\mu_m} = 1 + \frac{[\mu]}{\mu_m + \beta_m[\mu]} c_f + o(c_f), \quad (3.5a)$$

where $[k] = k_f - k_m$, $[\mu] = \mu_f - \mu_m$,

$$\alpha_m = \frac{3k_m}{3k_m + 4\mu_m} = \frac{1}{3}\frac{1 + \nu_m}{1 - \nu_m}, \qquad \beta_m = \frac{6}{5}\frac{k_m + 2\mu_m}{3k_m + 4\mu_m} = \frac{2}{15}\frac{4 - 5\nu_m}{1 - \nu_m}; \quad (3.5b)$$

$\nu_m$ is the Poisson ratio for the matrix.

The formula (3.4), especially its particular case (3.5), was arrived at by many authors on the base of various techniques (cf., e.g. [1], Chap. 2.2). For instance, the relation (3.4) was proposed in [24] as a simplest "one-particle" approximation, i.e. an approximation for the overall tensor of elastic moduli which relates the latter to the solution for a single inclusion embedded into an infinite matrix undergoing homogeneous strain at infinity.

The performed analysis enables us to conclude that the formula (3.4) reflects the overall properties of a dilute perfectly disordered suspension of spheres. It is noteworthy that the same conclusion can be reached on the basis of the Poisson–Wiener expansion too, since the function $R(\mathbf{y}_i - \mathbf{y}_j)$ is not present anywhere in the above analysis. In other words, eqn (3.4) is the exact overall tensor also for a dilute suspension of spheres whose centers form just a Poisson system of random points. On the other hand, if those centers do not form a Poisson system, the relations (3.4) and (3.5) are not in general valid, as it was shown in [11], when dealing with a suspension of spheres falling in pairs. This confirms the conclusion[25] that the statistics of the spheres' locations could affect even the first-order approximation, i.e. that for a dilute suspension of the overall tensor of elastic moduli $\mathbf{L}^*$.

## 4. SECOND-ORDER APPROXIMATION

To get a solution to the system (2.19) and (2.20) which holds to order $\gamma^2$, i.e. $c_f^2$, we should retain in the virial expansions (2.18) for $\mathbf{T}_1$, $\mathbf{T}_2$ the following terms:

$$\mathbf{T}_1(\mathbf{x}) = \mathbf{T}_{10}(\mathbf{x}) + \gamma\mathbf{T}_{11}(\mathbf{x}), \qquad \mathbf{T}_2(\mathbf{x}) = \mathbf{T}_{20}(\mathbf{x}). \tag{4.1}$$

Upon substituting eqn (4.1) into eqns (2.19) and (2.20), we obtain in an obvious manner the following equations for the three functions $\mathbf{T}_{10}$, $\mathbf{T}_{11}$ and $\mathbf{T}_{20}$

$$\nabla\cdot\{\mathbf{L}_m:\mathbf{Q}_{10}(\mathbf{x}) + h(\mathbf{x})[\mathbf{L}]:(\mathbf{E} + \mathbf{Q}_{10}(\mathbf{x}))\} = 0, \tag{4.2}$$

$$\nabla\cdot\{2(\mathbf{L}_m + (h(\mathbf{x}) + h(\mathbf{x} - \mathbf{z}))[\mathbf{L}]:\mathbf{Q}_{20}(\mathbf{x} - \mathbf{z}, \mathbf{x}) + [\mathbf{L}]:(\mathbf{Q}_{10}(\mathbf{x} - \mathbf{z})h(\mathbf{x}) + \mathbf{Q}_{10}(\mathbf{x})h(\mathbf{x} - \mathbf{z}))\} = 0, \tag{4.3}$$

$$\begin{aligned}\nabla\cdot\{(\mathbf{L}_m + h(\mathbf{x})[\mathbf{L}]):(\mathbf{Q}_{11}(\mathbf{x}) - \int \mathbf{Q}_{10}(\mathbf{x} - \mathbf{y})R(\mathbf{y})\,\mathrm{d}^3\mathbf{y})\\ - [\mathbf{L}]:\int \mathbf{Q}_{10}(\mathbf{x} - \mathbf{y})h(\mathbf{x} - \mathbf{y})R(\mathbf{y})\,\mathrm{d}^3\mathbf{y} - [\mathbf{L}]:\mathbf{E}F(\mathbf{x})\\ + (V_a - F(\mathbf{x}))[\mathbf{L}]:\mathbf{Q}_{10}(\mathbf{x}) + 2[\mathbf{L}]:\int \mathbf{Q}_{20}(\mathbf{x} - \mathbf{y}, \mathbf{x})h(\mathbf{x} - \mathbf{y})\,\mathrm{d}^3\mathbf{y}\} = 0,\end{aligned} \tag{4.4}$$

where

$$F(\mathbf{x}) = \int h(\mathbf{x} - \mathbf{y})\,R(\mathbf{y})\,\mathrm{d}^3\mathbf{y}, \tag{4.5}$$

$\mathbf{Q}_{nk}(\mathbf{x}) = \operatorname{def}\mathbf{T}_{nk}(\mathbf{x})$, $V_a = \frac{4}{3}\pi a^3$; it is acknowledged, when deriving eqn (4.4), that $\langle\mathbf{L}\rangle = \mathbf{L}_m + \gamma V_a\,[\mathbf{L}]$.

The eqn (4.2) is the same that already appeared in the first-order approximation, cf. eqn (3.2), and we assume its solution known. The eqn (4.3) can be rewritten, by virtue of eqn (4.2), in the form

$$\nabla\cdot\{(\mathbf{L}_m + (h(\mathbf{x}) + h(\mathbf{x} - \mathbf{z}))[\mathbf{L}]):(\mathbf{E} + \mathbf{Q}^{(2)}(\mathbf{x}; \mathbf{z}))\} = 0, \tag{4.6a}$$

where

$$\mathbf{Q}^{(2)}(\mathbf{x}; \mathbf{z}) = 2\mathbf{Q}_{20}(\mathbf{x} - \mathbf{z}, \mathbf{x}) + \mathbf{Q}_{10}(\mathbf{x}) + \mathbf{Q}_{10}(\mathbf{x} - \mathbf{z}). \tag{4.6b}$$

The eqn (4.6) is nothing but the equation for the disturbance $\mathbf{Q}^{(2)}(\mathbf{x};\mathbf{z})$ to the strain field in the infinite matrix, introduced by a pair of identical spherical inclusions whose centers are at the origin and at the point $\mathbf{z}$, when the strain tensor at infinity equals $\mathbf{E}$. Each of these inclusions, if it were alone, would disturb the strain field in the homogeneous matrix by $\mathbf{Q}_{10}(\mathbf{x})$ and $\mathbf{Q}_{10}(\mathbf{x}-\mathbf{z})$, respectively. Thus, the kernel $\mathbf{Q}_{20}(\mathbf{x}-\mathbf{z},\mathbf{x})$ is the field which should be added to the "single-inclusion" disturbances $\mathbf{Q}_{10}(\mathbf{x})$, $\mathbf{Q}_{10}(\mathbf{x}-\mathbf{z})$ in order to obtain the "double-inclusion" disturbance $\mathbf{Q}^{(2)}(\mathbf{x};\mathbf{z})$.

The displacement and strain fields in an infinite matrix containing two identical spherical inclusions have been recently constructed for an arbitrary homogeneous strain at infinity[26]. That is why we shall assume the kernel $\mathbf{T}_{20}(\mathbf{x}-\mathbf{z},\mathbf{x})$ and the strain field $\mathbf{Q}_{20}(\mathbf{x}-\mathbf{z},\mathbf{x})$ which it produces known. Then the last eqn (4.4) will describe the strain field in an infinite matrix containing a single spherical inclusion, which undergoes certain known body forces. In this way the full statistical solution to the problem (2.6)–(2.7) for the perfectly disordered suspension of spheres, which is correct to order $c_f^2$, can be obtained in a straightforward, though rather tedious, manner. We shall demonstrate this solution in detail elsewhere. However, the analysis simplifies considerably, if the calculation of the overall tensor of elastic moduli $\mathbf{L}^*$ up to order $c_f^2$ is our sole goal.

Indeed, due to eqns (2.16), (2.17) and (4.1), we have

$$\mathbf{L}^*:\mathbf{E} = \mathbf{L}_m:\mathbf{E} + c_f[\mathbf{L}]:\mathbf{A}(\mathbf{L}_m,\mathbf{L}_f):\mathbf{E} + \gamma^2[\mathbf{L}]:\int_{V_a} \operatorname{def}\mathbf{S}_1(\mathbf{x})\,\mathrm{d}^3\mathbf{x} + o(c_f^2), \quad (4.7)$$

where

$$\mathbf{S}_1(\mathbf{x}) = \mathbf{T}_{11}(\mathbf{x}) - \int \mathbf{T}_{10}(\mathbf{x}-\mathbf{y})R(\mathbf{y})\,\mathrm{d}^3\mathbf{y}. \quad (4.8)$$

The field $\mathbf{S}_1(\mathbf{x})$ is governed by the equation

$$\nabla\cdot\{\mathbf{L}_m:\operatorname{def}\mathbf{S}_1(\mathbf{x}) + h(\mathbf{x})[\mathbf{L}]:(\operatorname{def}\mathbf{S}_1(\mathbf{x}) - V_a\mathbf{A}:\mathbf{E})\} + \nabla\cdot\boldsymbol{\Omega} = 0,$$
$$\boldsymbol{\Omega} = [\mathbf{L}]:(\mathbf{A}:\mathbf{E})\,(V_a h(\mathbf{x}) - F(\mathbf{x})) + (V_a - F(\mathbf{x}))[\mathbf{L}]:\mathbf{Q}_{10}(\mathbf{x}) + 2[\mathbf{L}]:\mathbf{I}_{20}(\mathbf{x}), \quad (4.9)$$

which follows immediately from eqn (4.4), when acknowledging that $\mathbf{Q}_{10}(\mathbf{x}) = (\mathbf{A}-\mathbf{J}):\mathbf{E}$ at $|\mathbf{x}| \le a$, where $\mathbf{A} = \mathbf{A}(\mathbf{L}_m,\mathbf{L}_f)$ is the above introduced fourth-rank tensor, cf. eqn (3.3), and $\mathbf{J}$ is the unit fourth-rank tensor; also

$$\mathbf{I}_{20}(\mathbf{x}) = \int \mathbf{Q}_{20}(\mathbf{x}-\mathbf{y},\mathbf{x})\,h(\mathbf{x}-\mathbf{y})\,\mathrm{d}^3\mathbf{y}. \quad (4.10)$$

As seen from eqn (4.7), only the values of $\mathbf{S}_1(\mathbf{x})$ within the sphere $\mathbf{V}_a = \{\mathbf{x};\,|\mathbf{x}| \le a\}$ are needed. It turns out that those values could be found without solving the full-scale eqn (4.9), but this would need a bulky analysis that goes beyond the scope of this paper, whose primary objective is to display the performance of Volterra–Wiener series for random heterogeneous media. That is why we shall instead construct an approximate solution to eqn (4.9) within the sphere $\mathbf{V}_a$, which seems to be of particular interest.

With this aim in view, let us first note the following properties of the above introduced functions $F(\mathbf{x})$ and $\mathbf{I}_{20}(\mathbf{x})$:

$$F(\mathbf{x}) = V_a, \qquad \mathbf{I}_{20}(\mathbf{x}) = 0 \qquad \text{at } |\mathbf{x}| \le a; \quad (4.11\text{a})$$

$$\frac{\mathrm{d}}{\mathrm{d}n}F(\mathbf{x}) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}n}\mathbf{I}_{20}(\mathbf{x}) = 0 \qquad \text{at } |\mathbf{x}| = a, \quad (4.11\text{b})$$

with $\mathrm{d}/\mathrm{d}n$ denoting differentiation along the normal to the sphere $|\mathbf{x}| = a$. [For $\mathbf{I}_{20}(\mathbf{x})$, these properties follow from the condition (2.11), imposed on the kernel $\mathbf{T}_2$.] Keeping in mind eqn (4.11a) as well as the fact that $\mathbf{Q}_{10}(\mathbf{x})$ is constant at $|\mathbf{x}| \le a$, see Section 3, we infer now from eqn (4.9) that

$$\operatorname{def} \mathbf{S}_1(\mathbf{x}) = \mathbf{C}_1 \qquad \text{at} \quad |\mathbf{x}| \le a. \tag{4.12}$$

The constant tensor $\mathbf{C}_1$ could be specified by means of the conditions for continuity of the solution $\mathbf{S}_1(\mathbf{x})$ to eqn (4.9) and of the normal stress vector generated by $\mathbf{S}_1(\mathbf{x})$ at $|\mathbf{x}| = a$, with tensors of elastic moduli being $\mathbf{L}_f$ and $\mathbf{L}_m$ at $|\mathbf{x}| \le a$ and $|\mathbf{x}| > a$, respectively [the latter is an obvious consequence of the discontinuity of the coefficients in eqn (4.9) at $|\mathbf{x}| = a$].

The solution to eqn (4.9) can be represented in the form

$$\mathbf{S}_1(\mathbf{x}) = \mathbf{S}_1'(\mathbf{x}) + \mathbf{S}_1''(\mathbf{x}), \tag{4.13}$$

where $\mathbf{S}_1'$ and $\mathbf{S}_1''$ solve, respectively, the equations

$$\nabla\cdot\{\mathbf{L}_m : \operatorname{def} \mathbf{S}_1'(\mathbf{x}) + h(\mathbf{x})[\mathbf{L}]:(\operatorname{def} \mathbf{S}_1'(\mathbf{x}) - V_a\mathbf{A}:\mathbf{E})\} = 0; \tag{4.14}$$

$$\nabla\cdot\{\mathbf{L}_m : \operatorname{def} \mathbf{S}_1''(\mathbf{x}) + h(\mathbf{x})[\mathbf{L}]: \operatorname{def} \mathbf{S}_1''(\mathbf{x})\} + \nabla\cdot\boldsymbol{\Omega} = 0. \tag{4.15}$$

Obviously eqn (4.14) is the equation for the disturbance to the displacement field in the unbounded matrix, introduced by a single spherical inclusion, provided that the strain tensor at infinity equals $-V_a\mathbf{A}:\mathbf{E}$, cf. eqn (3.2). Consequently, due to eqn (3.3),

$$\operatorname{def} \mathbf{S}_1'(\mathbf{x}) = -V_a(\mathbf{A} - \mathbf{J}):(\mathbf{A}:\mathbf{E}) \qquad \text{at} \quad |\mathbf{x}| \le a. \tag{4.16}$$

The eqn (4.15) implies, in particular, that

$$\begin{aligned} \nabla\cdot\{\mathbf{L}_f : \operatorname{def} \mathbf{S}_1''(\mathbf{x})\} &= 0 \qquad \text{at} \quad |\mathbf{x}| < a, \\ \nabla\cdot\{\mathbf{L}_m : \operatorname{def} \mathbf{S}_1''(\mathbf{x})\} + \nabla\cdot\boldsymbol{\Omega} &= 0 \qquad \text{at} \quad |\mathbf{x}| > a. \end{aligned} \tag{4.17}$$

Due to eqns (4.12) and (4.16), the tensor $\operatorname{def} \mathbf{S}_1''(\mathbf{x})$ is constant at $|\mathbf{x}| \le a$; moreover, $\boldsymbol{\Omega}(\mathbf{x}) = 0$ at $|\mathbf{x}| \le a$, as seen from eqns (4.9) and (4.11). That is why the pair (4.17) can be replaced by a single equation over the whole $\mathbb{R}^3$, namely

$$\nabla\cdot\{\mathbf{L}_m : \operatorname{def} \mathbf{S}_1''(\mathbf{x})\} + \nabla\cdot\boldsymbol{\Omega} = 0. \tag{4.18}$$

In turn, a continuous and bounded solution to eqn (4.18) is given by the integral

$$\mathbf{S}_1''(\mathbf{x}) = \int \mathbf{G}^{(m)}(\mathbf{x} - \mathbf{y})\cdot\{\nabla\cdot\boldsymbol{\Omega}(\mathbf{y})\}\, d^3\mathbf{y}, \tag{4.19a}$$

where $\mathbf{G}^{(m)}$ is the displacement Green function for the unbounded matrix material. Thus

$$\operatorname{def} \mathbf{S}_1''(\mathbf{x}) = \int \boldsymbol{\Gamma}^{(m)}(\mathbf{x} - \mathbf{y}):\boldsymbol{\Omega}(\mathbf{y})\, d^3\mathbf{y}. \tag{4.19b}$$

Here $\boldsymbol{\Gamma}^{(m)}$ is the fourth-rank tensor field with the components

$$\Gamma^{(m)}_{ijkl} = \frac{1}{2}(G^{(m)}_{ik,lj} + G^{(m)}_{jk,li}).$$

One of the approximate methods successfully employed in the theory of heterogeneous materials, e.g. by Shermergor *et al.* (cf. [12], Chap. V, Section 8), is the so-called "singular approximation." The method is based on the well-known fact that the second derivative of the Green tensor for an elastic medium of arbitrary symmetry is a sum of a regular part which is an ordinary function, and of a singular part, proportional to the Dirac delta function

$$G^{(m)}_{ij,kl} = G^{(s)}_{ijkl}\delta(\mathbf{x}) + G^{(r)}_{ijkl}(\mathbf{x}).$$

The basic assumption of the singular approximation consists in neglecting the regular part of the latter derivative.

When employed to eqn (4.19b), this assumption yields def $\mathbf{S}_1''(\mathbf{x}) = 0$, $|\mathbf{x}| \le a$, since $\boldsymbol{\Omega}$ vanishes at $|\mathbf{x}| \le a$. Thus the tensor $\mathbf{C}_1$ coincides with the right-hand side of eqn (4.16). Inserting this value of $\mathbf{C}_1$ into eqn (4.7), we get for the overall tensor of elastic moduli

$$\mathbf{L}^* = \mathbf{L}_m + c_f[\mathbf{L}]:\mathbf{A} - c_f^2[\mathbf{L}]:\mathbf{A}:(\mathbf{A} - \mathbf{J}) + o(c_f^2). \tag{4.20}$$

Thus it turns out that for the given constitution of the medium the $c_f^2$ contribution to the overall moduli can be approximately calculated by means of the tensor $\mathbf{A} = \mathbf{A}(\mathbf{L}_m, \mathbf{L}_f)$, i.e. by summoning a solution (3.2) for the single inclusion problem. The interaction between the inclusions makes then appearance through influencing the far field of the strain tensor around each inclusion, which now equals $-V_a\mathbf{A}:\mathbf{E}$, cf. eqn (4.14). Such a possibility to account for the inclusion interaction in a composite material was proposed, e.g. by McCoy and Beran[25] in a particular case and later on, for arbitrary particulate materials, by Markov[24], who named it the method of effective field. Thus eqn (4.20) suggests that the overall properties for a perfectly disordered suspension of spheres in the singular approximation should be closely connected to those predicted by the method of effective field. This suggestion will be corroborated in the next section.

Consider now the case of isotropic constituents. Making use of the Eshelby results[23], we recast eqn (4.20) in the form

$$\begin{aligned} \frac{k^*}{k_m} &= 1 + \frac{[k]}{k_m + \alpha_m[k]}c_f\left(1 + \frac{\alpha_m[k]}{k_m + \alpha_m[k]}c_f\right) + o(c_f^2), \\ \frac{\mu^*}{\mu_m} &= 1 + \frac{[\mu]}{\mu_m + \beta_m[\mu]}c_f\left(1 + \frac{\beta_m[\mu]}{\mu_m + \beta_m[\mu]}c_f\right) + o(c_f^2), \end{aligned} \tag{4.21}$$

$\alpha_m$ and $\beta_m$ are given in eqn (3.5b). For rigid inclusions, $k_f = \mu_f = \infty$, and incompressible matrix, $k_m = \infty$, the shear modulus $\mu^*$ of the suspension under consideration thus becomes

$$\frac{\mu^*}{\mu_m} = 1 + 2.5\, c_f + 2.5\, c_f^2 + o(c_f^2). \tag{4.22}$$

The linear in $c_f$ term in eqn (4.22) is easily recognized as that from the famous Einstein relation. But the coefficient of the $c_f^2$ term is 2.5 and thus it differs essentially from the earlier obtained values (e.g. 155/32 [5], 5.01 [7], etc.). However, no conclusion can be drawn from such a discrepancy since our analysis of the overall moduli lost its rigor once the singular approximation was adopted.

In the next section we shall derive a more general formula for the overall tensor of elastic moduli, whose virial expansion will coincide with eqn (4.20) to the order $c_f^2$.

## 5. A HEURISTIC APPROACH

Let us assume that the displacement field in the suspension under consideration is given by a superposition of equishaped effects of the inclusions, that is,

$$\mathbf{u}(\mathbf{x}) = \mathbf{E}\cdot\mathbf{x} + \int \mathbf{T}_1(\mathbf{x} - \mathbf{y})\, C^{(1)}_\psi(\mathbf{y})\, d^3\mathbf{y}, \tag{5.1}$$

i.e. the higher-order terms are *a priori* neglected in the expansion (2.10). As shown in Sections 2 and 3, this assumption is consistent only if a linearization in $\gamma$ is simultaneously performed in the eqn (2.19). Such a linearization excludes $\mathbf{T}_2(\mathbf{x})$ and decisively simplifies eqn (2.19), cf. Section 3. The approach we propose here is to neglect $\mathbf{T}_2(\mathbf{x})$ from the very beginning, while retaining terms of order $o(\gamma)$ in eqn (2.19), generated by $\mathbf{T}_1(\mathbf{x})$. The obvious inconsistency of such an approach makes it a heuristic one. However, it has some spectacular consequences, as it will be seen below.

The eqn (2.19) with the kernel $\mathbf{T}_2(\mathbf{x})$ neglected can be recast to the form

$$\begin{aligned}\nabla\cdot\{\mathbf{L}_m : \operatorname{def}\mathbf{S}(\mathbf{x}) &+ h(\mathbf{x})[\mathbf{L}]:(\operatorname{def}\mathbf{S}(\mathbf{x}) + \mathbf{E})\\ &- \gamma F(\mathbf{x})[\mathbf{L}]:\mathbf{E} - \gamma[\mathbf{L}]: \int \operatorname{def}\mathbf{S}(\mathbf{x}-\mathbf{y})h(\mathbf{x}-\mathbf{y})R(\mathbf{y})\,\mathrm{d}^3\mathbf{y}\\ &+ \gamma(V_a - F(\mathbf{x}))[\mathbf{L}]: \operatorname{def}\mathbf{S}(\mathbf{x}) + \gamma^2[\mathbf{L}]:\mathbf{I}(\mathbf{x})\} = 0, \end{aligned} \tag{5.2}$$

where

$$\mathbf{I}(\mathbf{x}) = \iint h(\mathbf{x}-\mathbf{y}_1)\operatorname{def}\mathbf{T}_1(\mathbf{x}-\mathbf{y}_2)R(\mathbf{y}_1-\mathbf{y}_2)(1-R(\mathbf{y}_1))\,\mathrm{d}^3\mathbf{y}_1\,\mathrm{d}^3\mathbf{y}_2; \tag{5.3}$$

the functions $\mathbf{S}(\mathbf{x})$ and $F(\mathbf{x})$ are given in eqn (2.17) and (4.5), respectively.

A similar equation was considered in [10] for the heat conduction problem by means of power series expansion in $\gamma$. Guided by the above analysis of eqn (4.9), cf. Section 4, we employ here a simpler method, based upon the singular approximation.

The above introduced tensor field $\mathbf{I}(\mathbf{x})$ possesses the properties

$$\mathbf{I}(\mathbf{x}) = 0 \quad \text{at} \quad |\mathbf{x}| < a; \qquad \frac{\mathrm{d}}{\mathrm{d}n}\mathbf{I}(\mathbf{x}) = 0 \quad \text{at } |\mathbf{x}| = a, \tag{5.4}$$

as it readily follows from the definitions of the functions $h(\mathbf{x})$ and $R(\mathbf{x})$, cf. Section 2. Together with the similar properties (4.11) for $F(\mathbf{x})$, they yield

$$\operatorname{def}\mathbf{S}(\mathbf{x}) = \mathbf{C} \quad \text{at} \quad |\mathbf{x}| \le a. \tag{5.5}$$

The constant tensor $\mathbf{C}$ could be then specified by means of the same continuity conditions as those employed in Section 4.

In virtue of (5.5), the eqn (5.2) can be rewritten as

$$\begin{aligned}&\nabla\cdot\{\mathbf{L}_m: \operatorname{def}\mathbf{S}(\mathbf{x}) + h(\mathbf{x})[\mathbf{L}]:(\operatorname{def}\mathbf{S}(\mathbf{x}) + c_m\mathbf{E} - c_f\mathbf{C})\} + \nabla\cdot\tilde{\mathbf{\Omega}} = 0,\\ &\tilde{\mathbf{\Omega}} = \gamma[\mathbf{L}]:\{(\mathbf{E}+\mathbf{C})[V_a h(\mathbf{x}) - F(\mathbf{x})) + (V_a - F(\mathbf{x})]\operatorname{def}\mathbf{S}(\mathbf{x}) + \gamma\mathbf{I}(\mathbf{x})\}.\end{aligned} \tag{5.6}$$

Repeating the arguments which we have employed in Section 4 when dealing with the eqn (4.9), cf. eqns (4.13)–(4.16), we get that within the frame of the singular approximation the field $\tilde{\mathbf{\Omega}}$ will contribute nothing to the said continuity conditions, since $\tilde{\mathbf{\Omega}} = 0$ at $|\mathbf{x}| \le a$, and $\mathrm{d}\tilde{\mathbf{\Omega}}/\mathrm{d}n = 0$ at $|\mathbf{x}| = a$, cf. eqn (4.11) and (5.4). Thus to determine $\mathbf{C}$ it suffices to consider only the equation

$$\nabla\cdot\{\mathbf{L}_m: \operatorname{def}\mathbf{S}(\mathbf{x}) + h(\mathbf{x})[\mathbf{L}]:(\operatorname{def}\mathbf{S}(\mathbf{x}) + c_m\mathbf{E} - c_f\mathbf{C})\} = 0. \tag{5.7}$$

This is again the equation for the disturbance to the displacement field in the unbounded matrix, introduced by a single spherical inclusion, provided that the strain at infinity equals $c_m\mathbf{E} - c_f\mathbf{C}$. According to eqns (3.3) and (5.5), we thus obtain the following tensor equation for $\mathbf{C}$:

$$\mathbf{C} = (\mathbf{A} - \mathbf{J}):(c_m\mathbf{E} - c_f\mathbf{C});$$

$\mathbf{A} = \mathbf{A}(\mathbf{L}_m, \mathbf{L}_f)$, whose solution is obvious

$$\mathbf{C} = c_m (\mathbf{A} - \mathbf{J}):(c_m\mathbf{J} + c_f\mathbf{A})^{-1}:\mathbf{E}. \tag{5.8}$$

Inserting eqn (5.8) into eqn (2.16) we find that, within the frame of the proposed heuristic approach, the overall tensor of elastic moduli for the perfectly disordered suspension of spheres is

$$\mathbf{L}^* = \mathbf{L}_m + c_f[\mathbf{L}]:\mathbf{A}(\mathbf{L}_m, \mathbf{L}_f):(c_m\mathbf{J} + c_f\mathbf{A}(\mathbf{L}_m, \mathbf{L}_f))^{-1}. \tag{5.9}$$

The relation (5.9) was proposed by Markov[24] as another "one-particle" approximation—the "effective field" method. The basic assumption of that method lies in the supposition that the interaction between particles in a composite material makes appearance through influencing the averaged strain field around each particle due to the presence of the rest of the particles. Thus we can conclude that as far as the overall elastic properties are only concerned, the proposed heuristic approach is equivalent, within the frame of the singular approximation, to the effective field method.

Let us note now that, written to order $c_f^2$, the relation (5.9) becomes

$$\mathbf{L}^* = \mathbf{L}_m + c_f[\mathbf{L}]:\mathbf{A}:(\mathbf{J} - c_f(\mathbf{A} - \mathbf{J})) + o(c_f^2), \tag{5.10}$$

which coincides with eqn (4.20). Thus, the above obtained relation (4.20), found within the frame of the second-order approximation, is the truncated to the order $c_f^2$ virial expansion of the effective field relation (5.9) for the overall tensor of elastic moduli.

If both constituents are isotropic, the tensor $\mathbf{A}$, cf. eqn (3.5) when inserted into (5.9), yields for the overall bulk and shear moduli of the suspension

$$\frac{k^*}{k_m} = 1 + \frac{[k]\,c_f}{k_m + \alpha_m[k]c_m}, \qquad \frac{\mu^*}{\mu_m} = 1 + \frac{[\mu]\,c_f}{\mu_m + \beta_m[\mu]c_m}; \tag{5.11}$$

$\alpha_m$ and $\beta_m$ are given in (3.5b). The relations (5.11) were proposed by Levin[27]; they are closely connected with the Hashin–Shtrikman[28] and Walpole[29] bounds for the overall moduli of two-phase isotropic composite materials. Indeed, a simple analysis shows that the relations (5.11) coincide with the lower Hashin–Shtrikman bounds provided $[k][\mu] > 0$. If $[k][\mu] < 0$, the relation (5.11) for the bulk modulus coincides with the upper bound, and that for the shear modulus lies between the said bounds.

In the previous authors' work[10] the counterpart of eqn (5.9) for the overall thermal conductivity of a perfectly disordered suspension of spheres was derived; it appeared to be the known Maxwell formula—one of the Hashin–Shtrikman bounds for this case.

## 6. CONCLUDING REMARKS

In this paper we propose to employ the Volterra–Wiener functional expansion to statistical problems in elasticity of random heterogeneous materials. The approach seems to be novel and highly advantageous for this field; it offers unique possibilities when relating micro and macro properties of the materials. To demonstrate this we consider a simplified case of a random suspension of equisized spheres whose centers form the so-called perfect disorder of spheres (PDS). This is a particular configuration statistics which is of special interest because it seems to represent a case the most naturally occuring in application. It goes without saying that precise detailed information of the microstructure is however required to demonstrate that a given suspension of spheres can be properly modeled as a perfect disorder.

The sound physical ground of the PDS-field results in a very important feature of the Volterra–Wiener expansion, generated by this field, namely, it turns out to be virial with respect to the volume fraction $c_f$ of the spheres. For the kernels of this expansion an infinite hierarchy of conjugated equations is derived. [It is to be mentioned that the hierarchy could be considered as remotely akin to the known Bogoljubov–Born–Green–Kirkwood–Ivan (BBGKI) hierarchy in statistical physics of rarified gases with the volume fraction $c_f$ playing the role of the density parameter for the gase.] Due to the virial character of the said expansion, truncating the hierarchy after the $n$th-order term renders results which approximate to the order $c_f^n$ the averaged statistical characteristics for the random fields of displacement, strain, etc. The first- and the second-order approximations are considered in detail as an illustration, and the respective kernels, $\mathbf{T}_1$ and $\mathbf{T}_2$ are found. It appears that while $\mathbf{T}_1$ can be identified with the displacement field in an infinite body containing a single inclusion, the kernel $\mathbf{T}_2$ is closely connected, but not identical, to the displacement field in the body, containing two inclusions.

When dealing with overall elastic moduli, the outstanding position of the perfect disorder manifests itself again: For one thing, it appears that the determination of these moduli, at least to the order $c_f^2$, by means of the above-mentioned singular approximation of Shermergor *et al.*, cf. Section 4, can be easily performed without solving the full-scale equations for the second-order approximation. For another thing, the interaction between inclusions, within the frame of the same singular approximation, shows up through changing the strain far field around each inclusion; this is the earlier proposed idea of effective field.

Finally, it is important to note that unlike the known approach, initiated by Batchelor and Green[30] *et al.*, no difficulties, connected with convergence, normalization and so on, arise in our approach. This could be attached to the most appropriate grouping of statistical information in the expansion (2.10). Thus, the Volterra–Wiener approach accomplishes the suggestion of Chen and Acrivos that, "there may exist a method for calculating the effective bulk modulus which does not require a normalization to lead to an absolutely convergent integral and which gives, apparently, a different result" ([7], p. 349).

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